The Geometry Equations for Multiple Kinematic Structures
Using Direct Geometry

BELDEAN Călin¹, a, *, VUŞCAN Ioan¹, b, KACSO Kalman², c, PANC Nicolae¹, d

¹ Technical University of Cluj-Napoca, 103-105 Muncii Boulevard, Manufacturing Engineering
Department, Cluj-Napoca, Romania

² Technical University of Cluj-Napoca, 103-105 Muncii Boulevard, Department of Mechanical
Engineering, Cluj-Napoca, Romania

a calin.beldean@rompetrol.com, b givuscan@yahoo.com, c kacsokalman@gmail.com,
d nicolae.panc@tcm.utcluj.ro

Keywords: CNC general kinematic structure, CNC structure, kinematic structure.

Abstract. This paper presents two kinematic structures that are each able to move/rotate
along/around three axis simultaneously. These two structures are restricted to a common fixed bed
and have the possibility to perform a translational motion first and then rotate along or around each
axis. Next, the paper presents the equations used to describe the position and orientation of the first
structure in relation to the second. Lastly, a reduction of the general case is made to prove the
general mathematical model on a structure similar to a five axis turning and milling machine.

Introduction

Knowing the position and orientation of various kinematic structures is useful in many industry
situations, such as manufacturing by milling or turning, manipulation using robotic arms, cutting by
means of CNC controlled cutting machines, 3D measuring etc. These machines are made of one or
more kinematic structures, each having translation and/or rotation mechanisms. Whereas in the case
of robot manipulators or 3D measuring machines, there is only one kinematic structure present, in
the case of milling machines, there can be one or two kinematic structures working simultaneously,
like described in [1], or performing operations like in [2].

Taking into account the kinematics of turning and milling machines, and the possibility to move
or rotate along/around three to five axes, there is definitely a construction with two kinematic
structures that must work simultaneously. One kinematic structure can be considered the part
holding structure, and the other kinematic structure can be considered the tool holding structure,
both working simultaneously for the manufacturing of the finished part. Usually, these turning and
milling machines are left-oriented systems, therefore the positive direction of the axes is given
according to the left-hand rule.

Thus, two kinematic structures placed on the same frame, and able to perform six translations
and six rotations each, can mathematically describe the position and orientation of most types of
machines and, when reduced, they should be able to cover as many real industry situations as
possible. The goal is to have the most general mathematical description for the position and
orientation of each structure end in relation to the other.

Similar work was done by [3], [4] and [5] to describe the position and orientation of milling type
machines. The resulting equations can be used by computer software programmers to calculate the
tool’s and part future position knowing the part geometry.

General Mathematical Model

Two kinematic structures, K₁ and K₂, placed on the same frame, were drawn, Fig. 1, each able to
perform six translations and six rotations. This system is left-oriented, thus the positive direction of
the axes is given according to the left hand rule.
Similar to system 8 and 9, the origin of the systems 0≡1≡2, 4≡5≡6 and 11≡12 is identical.

The direct geometry equations described by [6] for all Eq. from (1) to (7) and (10) and (11) will be used to describe the position and orientation of each kinematic joint.

The first step is to set up the initial position vectors
\[
\mathbf{p}_{i-1} = \begin{bmatrix} p_{xi} & p_{yi} & p_{zi} \end{bmatrix}^T.
\]
(1)

and the axes along or around the motion are made
\[
\mathbf{k}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^T.
\]
(2)

and the constant value
\[
\Delta_i = \begin{Bmatrix} \{1; i = R\}; \{0; i = T\} \end{Bmatrix}.
\]
(3)

Using these values, the locating matrix \(T_{ii-1}\) between neighboring systems is to be written considering the fixed system origin.
\[
T_{ii-1} = T_{ii-1}^{(0)} \cdot T_{\Delta}(k_i; q_i).
\]
(4)

where:

Fig. 1 – General kinematic structure able to perform six translations and six rotations

Similar to system 8 and 9, the origin of the systems 0≡1≡2, 4≡5≡6 and 11≡12 is identical.

Applied Mechanics and Materials Vol. 808
\[ T_{i_{ii-1}}^{(0)} = \begin{bmatrix} R_{i_{ii-1}} & i_{ii-1}^{(0)} \\ p_{i_{ii-1}} & 1 \end{bmatrix} , \text{ where } R_{i_{ii-1}} = I_3. \] 

\[ T_\Delta \left( \vec{k}_i; q_i \right) = \begin{bmatrix} R(\vec{k}_i; q_i \cdot \Delta_i) & i_{ii-1}^{(0)} \\ p_{ii-1} + (1 - \Delta_i) \cdot q_i \cdot \vec{k}_i \end{bmatrix}. \] 

The parameter \( R(\vec{k}_i; q_i \cdot \Delta_i) \) is equal to \( I_3 \) if the movement is a translation, or is the rotation matrix if the movement is rotation.

Having the locating matrices of the neighboring systems the matrixes of each system can be written considering the fixed system origin.

\[ T_{i0} = \prod_{j=2}^{i} T_{jj-1} \] 

\[
T_{i0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad T_{20} = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad T_{30} = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & q_1 \\ 0 & 0 & 1 & d_4 + q_3 \end{bmatrix} ; 
\]

\[
T_{40} = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & \cos(q_4) & \sin(q_4) & q_1 \\ 0 & -\sin(q_4) & \cos(q_4) & d_4 + d_5 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; 
\]

\[
T_{50} = \begin{bmatrix} \cos(q_5) & 0 & -\sin(q_5) & q_2 \\ \sin(q_4) \cdot \sin(q_5) & \cos(q_4) \cdot \cos(q_5) & \sin(q_4) & q_1 \\ \cos(q_4) \cdot \sin(q_5) & -\sin(q_4) \cdot \cos(q_5) & \cos(q_4) \cdot \cos(q_5) & d_4 + d_5 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; 
\]

\[
T_{60} = \begin{bmatrix} \cos(q_5) \cdot \cos(q_6) & \cos(q_5) \cdot \sin(q_6) & -\sin(q_5) & q_2 \\ \cos(q_6) \cdot \sin(q_5) & \cos(q_6) \cdot \cos(q_5) & \cos(q_5) \cdot \sin(q_4) & q_1 \\ -\sin(q_5) \cdot \cos(q_4) & \cos(q_5) \cdot \cos(q_6) & \cos(q_5) \cdot \cos(q_6) \cdot \sin(q_5) & d_4 + d_5 + q_3 \\ \sin(q_5) \cdot \sin(q_6) & +\cos(q_4) \cdot \cos(q_6) \cdot \cos(q_5) - \cos(q_4) \cdot \sin(q_5) \cdot \sin(q_6) & \cos(q_4) \cdot \sin(q_6) & d_4 + d_5 + q_3 \end{bmatrix} . 
\]

\[
T_{70} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & d_4 + q_7 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad T_{80} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_1 + b_2 + q_8 \\ 0 & 0 & 1 & d_4 + q_7 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad T_{90} = \begin{bmatrix} 1 & 0 & 0 & q_9 \\ 0 & 1 & 0 & b_1 + b_2 + q_8 \\ 0 & 0 & 1 & d_4 + q_7 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; 
\]

\[
T_{100} = \begin{bmatrix} \cos(q_{10}) & 0 & -\sin(q_{10}) & q_9 \\ \sin(q_{10}) & 0 & \cos(q_{10}) & d_4 + q_7 \\ 0 & 1 & 0 & b_1 + b_2 - b_3 + q_8 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; 
\]
With the position and orientation defined for each kinematic element (8) and (9), the position and orientation of each system end relatively to the other system end, $T_{\text{Tool - Part}}$ and $T_{\text{Part - Tool}}$, can be calculated with Eq. (10) and (11).

$$T_{\text{P-T}}^{-1} = T_{\text{60}}^{-1} \cdot T_{\text{120}}.$$  \hspace{1cm} (10)$$

$$T_{\text{T-P}}^{-1} = T_{\text{120}}^{-1} \cdot T_{\text{60}}.$$  \hspace{1cm} (11)$$

Due to space limitations the matrices for $T_{\text{Tool - Part}}$ and $T_{\text{Part - Tool}}$ will not be edited here, but applied to the case below.

**Case Study**

With position and orientation defined and described in (10) and (11), the reduction to specific cases will be done by canceling the kinematic joint where there is no movement and by canceling unnecessary distances.

Fig. 2 contains the draft of this case, showing a kinematic structure that is the result of canceling some of the kinematic joints of the general case shown in Fig. 1. The canceled joints are $q_1$, $q_2$, $q_3$, $q_4$, $q_5$, $q_{11}$ and $q_{12}$ thus only five out of twelve joints remain. This structure is similar to the kinematic structure of a turning and milling machine similar to those shown in [7].

Considering the new kinematic structure, matrices (8) and (9) can be rewritten given the fixed system origin.

$$T_{\text{60}} = \begin{pmatrix}
\cos(q_6) & \sin(q_6) & 0 & 0 \\
-\sin(q_6) & \cos(q_6) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \hspace{1cm} (12)$$

$$T_{\text{120}} = \begin{pmatrix}
\cos(q_{10}) & 0 & -\sin(q_{10}) & q_9 \\
0 & 1 & 0 & b_1 + b_2 - b_3 - b_4 + q_8 \\
\sin(q_{10}) & 0 & \cos(q_{10}) & d_4 + q_7 \\
0 & 0 & 0 & 1
\end{pmatrix}. \hspace{1cm} (13)$$

Having the position and orientation defined for the end of each kinematic structure (12) and (13), the position and orientation of each system end relative to the other system end, $T_{\text{T-P}}$ and $T_{\text{P-T}}$, can be calculated with Eq. (10) and (11).
Finally, Eq. (14) describes the position and orientation of the tool in relation to the manufactured part and Eq. (15) describes the position and orientation of the manufactured part in relation to the tool. Parameters $q_6$ and $q_{10}$ are rotations and are determined using only data from the geometry of the part where $q_9$, $q_8$ and $q_7$, meaning X, Y and Z, are calculated using data from geometry in relation to $q_6$. 

$$T_{T-P} = \begin{bmatrix}
\cos(q_6) \cdot \cos(q_{10}) & -\sin(q_6) & -\cos(q_6) \cdot \sin(q_{10}) & \cos(q_6) \cdot q_6 - \sin(q_6) \cdot (b_1 + b_2 - b_3 + q_8) \\
\cos(q_{10}) \cdot \sin(q_6) & \cos(q_6) & -\sin(q_6) \cdot \sin(q_{10}) & \sin(q_6) \cdot q_9 + \cos(q_6) \cdot (b_1 + b_2 - b_3 + q_8) \\
\sin(q_{10}) & 0 & \cos(q_{10}) & d_4 + q_7 \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{14}$$

$$T_{P-T} = \begin{bmatrix}
\cos(q_6) \cdot \cos(q_{10}) & \cos(q_{10}) \cdot \sin(q_6) & \sin(q_{10}) & -d_4 \cdot \sin(q_{10}) - q_0 \cdot \cos(q_{10}) \\
-\sin(q_6) & \cos(q_6) & 0 & -q_7 \cdot \sin(q_{10}) \\
-\cos(q_6) \cdot \sin(q_{10}) & -\sin(q_6) \cdot \sin(q_{10}) & \cos(q_{10}) & b_3 - b_2 - b_1 - q_8 \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{15}$$
Conclusions

The kinematic structure shown in Fig. 1 was designed as a generalized structure. When reduced, this structure is meant to cover as many real industry situations as possible.

The position and orientation of one end in relation to the other was defined using the direct model described in [6].

One case was presented, Fig. 2, to prove the mathematical model when reduced from a structure with six translations and six rotations to one with three translations and two rotations. This new structure is the kinematic structure of a five axis turning and milling machine. This case can be replicated and from the general structure after reduction to have from two axis turning machine to five axis turning and milling machine and from three axis milling machine to five axis milling machine. In the same way can be obtained the kinematic structure and set up the position and orientation of measuring machines, measuring arms, robot manipulators etc.

Having the geometry of the finished part and the kinematic structure equations, computer software can use this data to aid engineers in real life engineering situations.

References


Modern Technologies in Manufacturing
10.4028/www.scientific.net/AMM.808

The Geometry Equations for Multiple Kinematic Structures Using Direct Geometry
10.4028/www.scientific.net/AMM.808.292